Name KEY

MTH 252 Midterm Exam

You may use a calculator on this exam, but you should also show work if you hope to earn any partial credit for an incorrect answer (and for some problems, you're required to show your work). Point values are listed by each problem for a total of 90 points. **For problems where you show work, please circle or box your final answers.** If you are unsure about what a question is asking, raise your hand or come up to my desk. Good luck!

1. (6 pts) Use l'Hospital's rule to evaluate the following limit, showing your work: $\lim_{x\to 1} \frac{\ln x}{x-1}$

Note: since $\ln(1) = 1 - 1 = 0$, we can use l'Hospital's rule

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ & $\frac{d}{dx}(x-1) = 1$

so we have $\lim_{x\to 1} \frac{\frac{1}{x}}{1} = \boxed{1}$

2. (10 pts) Evaluate each of the following indefinite integrals.

a) $\int 3x^3 - \frac{4}{\sqrt{x}} dx = \int 3x^3 - 4x^{-1/2}\, dx$

$$= \frac{3x^4}{4} - \frac{4x^{1/2}}{1/2} + C = \boxed{\frac{3}{4}x^4 - 8x^{1/2} + C}$$

b) $\int 2 + 5\sec^2\theta\ d\theta$

$$\boxed{2\theta + 5\tan\theta + C}$$

3. (6 pts) Find a function f that satisfies $f'(x) = 2x + 4$ and $f(1) = 8$.

$f(x) = x^2 + 4x + C$

$f(1) = 1^2 + 4(1) + C = 8$

$5 + C = 8$

$C = 3$

$$\boxed{f(x) = x^2 + 4x + 3}$$

4. (8 pts) Consider a sports car that accelerates from 0 ft/sec to 88 ft/sec in 5 seconds (note that 88 ft/sec is about 60 mph). The car's velocity is given in the table below. You may assume that the velocity is an increasing function here.

t	0	1	2	3	4	5
$v(t)$	0	30	52	68	80	88

Find lower and upper estimates for the distance the car travels in those 5 seconds.

left-hand sum: $1(0)+1(30)+1(52)+1(68)+1(80)=230$

right-hand sum: $1(30)+1(52)+1(68)+1(80)+1(88)=318$

Lower estimate 230 ft

Upper estimate 318 ft

5. (8 pts) Find the area of the region in the plane enclosed by the x-axis and the graph of $f(x) = x^2 - 16$.

x-intercepts : ± 4

$\int_{-4}^{4} x^2-16\, dx = \frac{x^3}{3} - 16x \Big|_{-4}^{4} = \left(\frac{4^3}{3} - 16(4)\right) - \left(\frac{(-4)^3}{3} - 16(-4)\right)$

$= -\frac{256}{3}$ so the area is $\boxed{\frac{256}{3}}$

6. (6 pts) The rate of change of a population of beetles in a forest is given by the function $p(t)$, where the units for t are years since 1970, and the units for $p(t)$ are in thousands of beetles per year. Interpret the following equation as a statement about the population; be as specific as possible, and answer with a sentence.

$\int_{10}^{30} p(t)\, dt = -6$

integral of a rate of change = net change

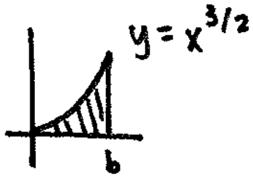
$t=10 \rightarrow 1980 \quad t=30 \rightarrow 2000$

Between 1980 and 2000, the population of beetles in a forest suffered a net decrease of 6000 beetles.

$$\int_0^b x^{3/2}\,dx$$

$\uparrow$

7. (10 pts) [The area between $y = x^{3/2}$ and the x-axis, for $0 \le x \le b$] is $\frac{64}{5}$. Find the value of b using the Fundamental Theorem of Calculus.



$$\int_0^b x^{3/2}\,dx = \frac{64}{5}$$

$$\left.\frac{x^{5/2}}{\frac{5}{2}}\right|_0^b = \frac{64}{5}$$

$$\frac{2}{5}b^{5/2} = \frac{64}{5}$$

$$\cdot\frac{5}{2} \qquad \cdot\frac{5}{2}$$

$$\to b^{5/2} = 32$$

$$\boxed{b=4}$$

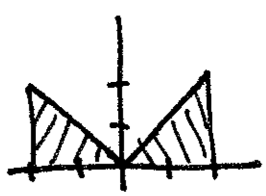
8. (6 pts) Find the derivative: $\frac{d}{dx}\left(\int_1^{x^2} \cos(t^3)\ dt\right)$

$g(x) = \int_1^x \cos(t^3)\,dt$ is an antiderivative of $f(x) = \cos(x^3)$... that is, $g' = f$

we want $\frac{d}{dx}(g(x^2)) = g'(x^2)\cdot 2x = f(x^2)\cdot 2x$

$= \cos((x^2)^3)\cdot 2x = \boxed{2x\cos(x^6)}$

9. (6 pts) Find $\int_{-2}^{2} |x|\ dx$ geometrically—that is, by taking advantage of the fact that the area under the curve is made up of simple shapes.



each triangle has area

$\frac{1}{2}bh = \frac{1}{2}\cdot 2\cdot 2 = 2$

so $\int_{-2}^{2} |x|\,dx = \boxed{4}$

10. (8 pts) Evaluate the indefinite integral using the substitution rule, and show your work:

$\int 6y(y^2+6)^3\, dy$

$u = y^2 + 6$

$du = 2y\, dy$

$\cdot 3 \quad \cdot 3$

$3\,du = 6y\, dy$

so we have:

$\int u^3 \cdot 3\, du$

$= \frac{3u^4}{4} + C$

$= \boxed{\frac{3(y^2+6)^4}{4} + C}$

11. (6 pts) The velocity of a particle after t seconds is given by $v(t) = \sqrt{16 - 4t^2}$, in ft/sec.

a) Find the particle's displacement over the time interval [0, 2]. You may use your calculator to evaluate the integral. Round to 3 places and include units.

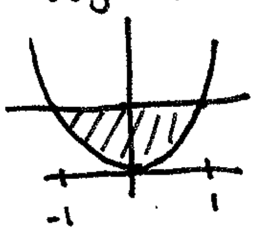
$\int_0^2 \sqrt{16-4t^2}\, dt \approx \boxed{6.283 \text{ ft}}$

b) Would the total distance traveled over this time period be the same as displacement here? Explain. (Answering this question does **not** require calculating the distance traveled.)

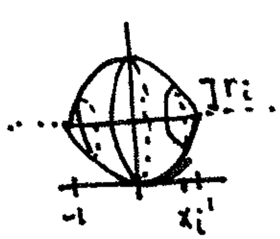
Yes. The velocity is positive over the whole time interval, so the particle does not change direction (which would be the only case where distance traveled $\neq$ displacement).

12. (10 pts) Find the volume of the solid obtained by revolving the region enclosed by $y = 1$ and $y = x^2$ about the line $y = 1$. Show your work.

region



solid



vertical slices are discs

volume of slice:

$\pi r_i^2 \Delta x$ but $r_i = 1 - x_i^2$

so $\pi(1-x_i^2)^2 \Delta x$

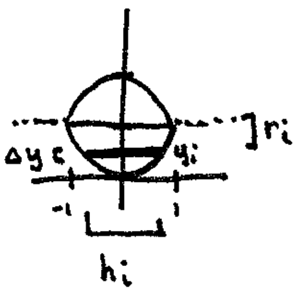
volume of solid:

$\int_{-1}^{1} \pi(1-x^2)^2\, dx = \int_{-1}^{1} \pi(1-2x^2+x^4)\, dx$

$= \pi\left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_{-1}^{1}$

$= \pi\left[\left(1-\frac{2}{3}+\frac{1}{5}\right) - \left(-1+\frac{2}{3}-\frac{1}{5}\right)\right] = \boxed{\frac{16\pi}{15}}$

Bonus (3 pts): Verify your answer to the last problem using the method of cylindrical shells, and show your work.



volume of shell:

$2\pi r_i \cdot h_i \cdot \Delta y$

but $r_i = 1 - y_i$

& since $y = x^2$ implies $x = \pm\sqrt{y}$,

$h_i = 2\sqrt{y_i}$

so we have $2\pi(1-y_i)(2\sqrt{y_i})\Delta y$

volume of solid:

$$\int_0^1 4\pi(1-y)(\sqrt{y})\,dy$$

$$= 4\pi \int_0^1 y^{1/2} - y^{3/2}\,dy$$

$$= 4\pi\left(\frac{y^{3/2}}{3/2} - \frac{y^{5/2}}{5/2}\right)\Big|_0^1$$

$$= 4\pi\left(\frac{2}{3} - \frac{2}{5}\right)$$

$$= 4\pi\left(\frac{4}{15}\right)$$

$$= \boxed{\frac{16\pi}{15}}$$