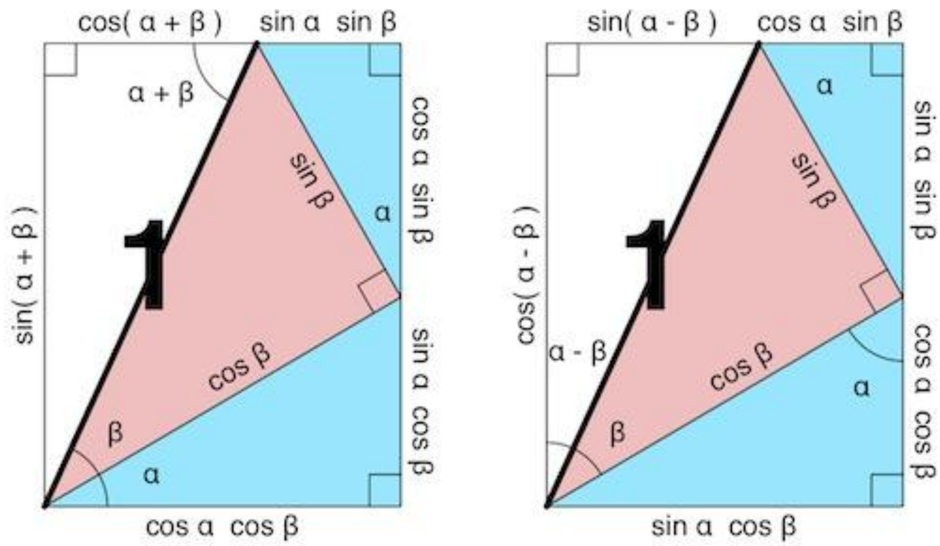
Deriving the Sum and Difference Formulas



Why are the diagrams labeled like that?

Once α, β, and the side length "1" are labeled in each diagram, every other label in the diagram can be found using right triangle trigonometry. For example, look at the pink right triangle in the diagram on the left. It has an angle β and a hypotenuse 1. So we know that $\sin\beta = \frac{opposite}{1}$, so $opposite = \sin\beta$.

Therefore, the opposite side of that right triangle is labeled $\sin\beta$. We can label $\cos\beta$ similarly. Figuring out the side lengths of the other triangles in the picture involves calculating some additional angle measures. Since the side labeled "1" in the left diagram cuts through two parallel lines (the top and bottom of the rectangle), we know that the angle between the "1 side" and the top of the rectangle is the same as between the "1 side" and the bottom of the triangle—that is, $\alpha + \beta$. Once that angle is labeled, we can figure out the side lengths of the white triangle.

Finally, why is the angle in the top blue triangle also labeled α? In the bottom blue triangle, we have angles of α and 90°, so the third angle must be (90 – α)°. Then we see that (90 – α)°, plus the 90° angle in the pink triangle, plus that angle in the top blue triangle must add to 180°. So that angle must be α.

In that triangle, we see that $\sin\alpha = \frac{opposite}{\sin\beta}$ so $opposite = \sin\alpha\sin\beta$ and is labeled as such. Similar processes allow us to label the rest of the line segments. The diagram on the right follows the same process' the angles are just labeled differently.

How do the diagrams give us the formulas?

The opposite sides of a rectangle have the same length. So the diagram on the left tells us $\mathbf{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$ and $\cos(\alpha+\beta) + \sin\alpha\sin\beta = \cos\alpha\cos\beta$. Subtracting the $\sin\alpha\sin\beta$ term from both sides gives us $\mathbf{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.}$ Similarly, the diagram on the right gives us $\cos(\alpha-\beta)$ directly and $\sin(\alpha-\beta)$ with just a bit of algebra.