Section 4: Solving Systems of Linear Equations

Introductory Summary:

We've solved systems of linear equations before – here's a classic one from Algebra I:

$$\begin{aligned} x + y &= 7 \\ x - y &= 5 \end{aligned}$$

In previous classes, you've probably learned several different ways to solve this system: guessing and checking, graphing to find the intersection, substitution, or addition/elimination. All of those methods work well if the system is small and simple. But, what if you had to find the solution of this system of linear equations?

$$\begin{aligned} 3w + 2x - 5y + 1z &= -7 \\ 1w + 1x + 1y - 10z &= -66 \\ 7w - 5x + 4y - 1z &= -12 \\ -2w + 4x - 6y + 5z &= 27 \end{aligned}$$

Do you want to solve this by hand? I'll give you a hint – the answers are single digit integers.....

Well, we could solve this by hand, but it would take a lot of time. Let's try some calculator magic instead:

Step 1: Line up the equations so that each variable is in one column; and so the equals sign lines up on the right.

our example: already done!

Step 2: Enter a matrix in your calculator – each row is all the numbers (but not the variables) from each equation. Use zeros for any missing terms.

our example:

This line takes the place of the equals signs. It's called the augmentation line – the calculator does not need to see it, but humans should always include it.

$$[A] = \left[\begin{array}{cccc:c} 3 & 2 & -5 & 1 & -7 \\ 1 & 1 & 1 & -10 & -66 \\ 7 & -5 & 4 & -1 & -12 \\ -2 & 4 & -6 & 5 & 27 \end{array}\right]$$

Step 3: Find "rref" on your $\boxed{\text{Matrix}}$ $\boxed{\text{Math}}$ menu on the calculator. Apply it to your matrix.

our example:

rref([A]) press $\boxed{\text{ENTER}}$

..... so the calculator does some magic, and we get ...

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right]$$

Step 4: This matrix shows equations that are equivalent (have the same solutions) as our original equations. Put the variables back in:

$$\begin{aligned} 1w + 0x + 0y + 0z &= -1 \\ 0w + 1x + 0y + 0z &= 2 \\ 0w + 0x + 1y + 0z &= 3 \\ 0w + 0x + 0y + 1z &= 7 \end{aligned}$$

or, to make it easier to read:

$$\begin{array}{ccccc} w & & & & = -1 \\ & x & & & = 2 \\ & & y & & = 3 \\ & & & z & = 7 \end{array}$$

Check it – that's the correct solution!

Try the process again with the simpler system:

Step 1: Line up the variables.

$$\begin{aligned} x + y &= 7 \\ x - y &= 5 \end{aligned}$$

Step 2: Put the augmented matrix into the calculator:

$$[B] = \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & -1 & 5 \end{array}\right]$$

Step 3: Ask for reduced row echelon form:

rref([B])

Step 4: Read the answer:

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 1 \end{array}\right]$$

$$\begin{aligned} x \quad\quad &= 6 \\ y &= 1 \end{aligned}$$

cool!

Some questions we're leading up to:

1) How the heck does the calculator do that? It uses Gauss-Jordan elimination, which we'll be learning soon. It's a technique that grew out of the addition/elimination method we learned in Algebra I.

2) What happens if there's no solution? That's an insomnia question for now!

3) What happens if there's more than one solution? This is fun – we'll be able to describe infinite solution sets using the vector descriptions of lines and planes we did in Homework ~~15~~. We'll be able to read those vector descriptions almost directly from the reduced row echelon form of the matrix.

4) What is this good for? Ah-ha, the classic question! Systems of linear equations are used in many applications, from business, to the life sciences, to geology and crystalography, and future mathematics. You'll see them a lot next year in differential equations.

rref by hand = Gauss-Jordan Elimination

Remember what we just did -- we learned how to solve systems of equations by rewriting the system of equations as a matrix of coefficients, then we ask the calculator to "rref" that matrix, then we interpret the simplified matrix to get the answer to the original system of equations.

Let's review the simple example:

$$\begin{matrix} x+y=7 \\ x-y=5 \end{matrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & -1 & 5 \end{array}\right] \overset{rref}{\Rightarrow} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{matrix} 1x+0y=6 \\ 0x+1y=1 \end{matrix} \Rightarrow \begin{matrix} x=6 \\ y=1 \end{matrix}$$

What does rref mean, and how could you do it by hand?

rref stands for "reduced row-echelon form." It means that we want to get the matrix so that the left side looks as much as possible like the identity matrix -- with 1's on the main diagonal, and 0's everywhere else.

We get rref by using these legal moves:

- Rule 1. Allowed to switch any two rows.
- Rule 2. Allowed to mulitiply (or divide) any row by any non-zero number.
- Rule 3. Allowed to add (or subtract) any multiple of one row to any other row.

(These rules are actually exactly the same rules as in the old algebra 1 "addition/elimination" method of solving systems of equations. In fact, that's why rref works, because it is the addition/elimination method of solving equations, but we're being too lazy to keep writing down the variables.)

We also need an overall plan for getting rref:

1. Get a 1 in the upper left corner.
2. Get 0's in the rest of the first column.
3. Get a 1 in the next position on the main diagonal.
4. Get 0's in the rest of that column.
5. Repeat steps 3 and 4 as needed.

It is important to do these steps in order -- for instance, if you don't clear the first column before moving on to the second, your manipulations on the second column will mess up the numbers in first column.

Let's do our example from above:

$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & -1 & 5 \end{array}\right]$ subtract row 1 from row 2 (or, $r_2 - r_1 \to r_2$)

$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -2 & -2 \end{array}\right]$ divide row 2 by -2 (or, $r_2 \div -2 \to r_2$)

$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 1 \end{array}\right]$ subtract row 2 from row 1 (or, $r_1 - r_2 \to r_1$)

$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 1 \end{array}\right]$ done!

One more example:

Start with the system of equations $\begin{matrix} 2x+3y=8 \\ x+y=3 \end{matrix}$ and put it in matrix form. Then find "reduced row-echelon form" of that matrix:

$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 1 & 1 & 3 \end{array}\right]$ switch row 1 and row 2 ($r_2 \leftrightarrow r_1$)

$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 3 & 8 \end{array}\right]$ subtract twice row 1 from row 2 ($r_2 - 2r_1 \to r_2$)

$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right]$ subtract row 2 from row 1 ($r_1 - r_2 \to r_1$)

$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$ done!

So what's the solution to the system? $\begin{matrix} x=1 \\ y=2 \end{matrix}$

Let's check the solution:

$$\begin{matrix} 2x+3y=8 \\ x+y=3 \end{matrix} \Rightarrow \begin{matrix} 2(1)+3(2)\overset{?}{=}8 \\ (1)+(2)\overset{?}{=}3 \end{matrix} \Rightarrow \begin{matrix} 8=8 \\ 3=3 \end{matrix}$$

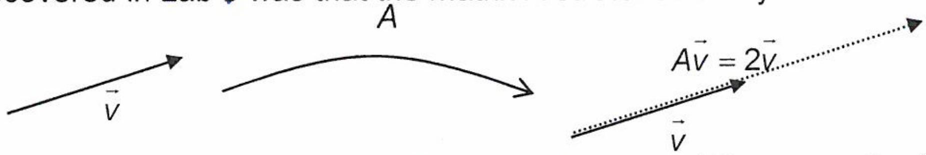
yes!

Introduction to Linear Transformations
-- What does matrix multiplication do to vectors?

Let's review what we found out in Lab ~~8~~ B, where we explored what multiplying by a particular matrix does to vectors in $\mathbb{R}^2$. For instance, we started with this matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

What we discovered in Lab ~~8~~ B was that the matrix A stretches every vector in $\mathbb{R}^2$ by 2.



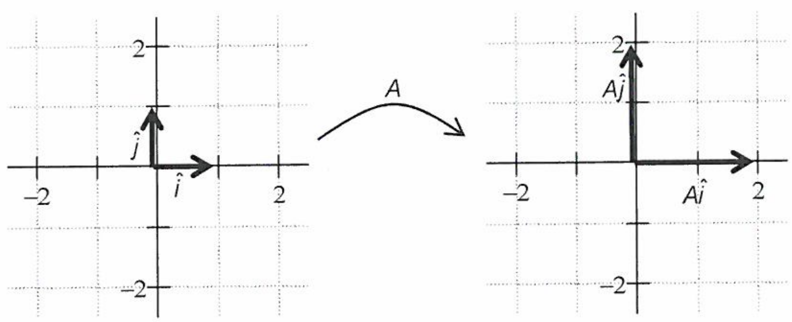
We discovered this by analyzing several different vectors, and then guessing that A would do the same thing to all vectors -- but there's actually a more efficient way to do the analysis. Once we know what the matrix does to the basis vectors $\hat{i}$ and $\hat{j}$, we know what the matrix will do to <u>every</u> vector. Here's how:

Suppose $\vec{v} = a\hat{i} + b\hat{j}$. Then

$$\left.\begin{aligned} A\vec{v} &= A\left(a\hat{i} + b\hat{j}\right) \\ &= A(a\hat{i}) + A(b\hat{j}) \\ &= a(A\hat{i}) + b(A\hat{j}) \end{aligned}\right\}$$

This statement is actually a big deal -- more news later!

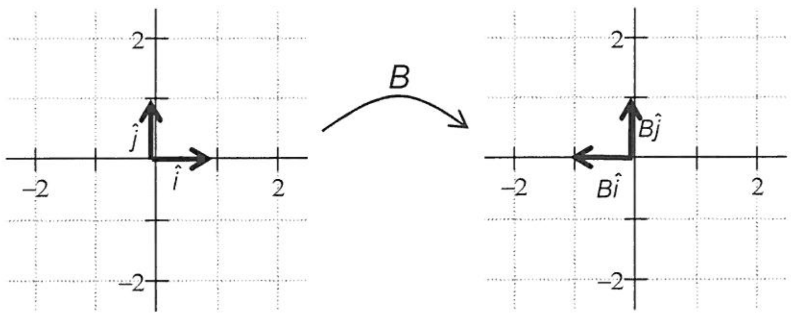
In our example, $A\hat{i} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $A\hat{j} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.



So, we look at these pictures and know that matrix A stretches the x-component and y-component of every vector by 2 -- so it stretches every vector by 2.

What about matrix B?

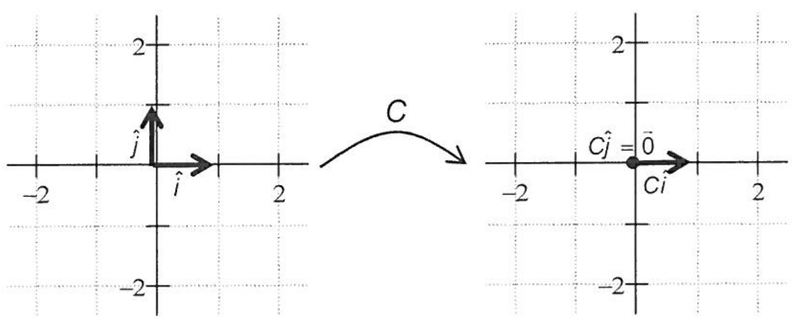
In our example, $B\hat{i} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $B\hat{j} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



So, we look at these pictures and know that matrix B switches the x-component to the negative side, and leaves the y-component of every vector the same -- so it reflects every vector across the y-axis.

What about matrix C?

In our example, $C\hat{i} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $C\hat{j} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



So, we look at these pictures and know that matrix C keeps the x-component the same, and smashes the y-component of every vector down to 0 -- so it projects every vector onto the x-axis.

Matrix Inverses and Their Use in Solving Systems of Equations

Matrix Inverses:

You may have noticed that when we did matrix arithmetic, we did addition, subtraction, and multiplication, but we never discussed division. Why was that? Well, "division" of matrices is trickier than the other operations. In fact, we're not really going to divide matrices so much as find their multiplicative inverses.

> Aside: What's a multiplicative inverse? In numbers, it's the reciprocal – the number that you would multiply by to get back to 1 (the multiplicative identity).
>
> example: $3 \cdot \frac{1}{3} = 1$
>
> So, $\frac{1}{3}$ is the multiplicative inverse of 3. And, in place of dividing by 3, we can always multiply by $\frac{1}{3}$.

So, what we need for any matrix A is a matrix called A^{-1} ("A-inverse") such that $A \cdot A^{-1} = \mathrm{I} = A^{-1} \cdot A$.

Here's an example:

In Lab 8 we looked at the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The inverse of matrix A is the matrix $A^{-1} = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$.

Let's check:

$$A \cdot A^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1} \cdot A = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Aside: By the way, this one should make sense in terms of linear transformations. Matrix A stretches every vector twice as long, matrix A^{-1} will shrink every vector to half its length. If you do both, you return to the original vectors.

Well, that wasn't so hard – what about another matrix?

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

<u>Our question</u>: What is B^{-1}?

What matrix times B equals $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

<u>Method 1:</u> (calculator magic)

- enter B in your calculator as a matrix.
- ask for $[B]^{-1}$. (use the $\boxed{x^{-1}}$ key on TI-83's, on the TI-89's, use B^-1)
- calculator gives:

$$B^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

weird! check that B^{-1} is correct by multiplying it with B.

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix} = ?$

<u>Method 2</u>: (brute force)

$$B \cdot B^{-1} = \mathrm{I}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which gives us four equations with four unknowns[2]...

$$\begin{array}{ccccc} a & & +2c & & =1 \\ & b & & +2d & =0 \\ 3a & & +4c & & =0 \\ & 3b & & +4d & =1 \end{array}$$

which we can solve as a system of equations, and get...

$a=-2 \quad b=1 \quad c=1.5 \quad d=-.5$

$$\Rightarrow \quad B^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix}$$

yuck! that was too hard!

[2] ok, in this case it's really two systems, each with two equations and two unknowns.

Method 3: (way cool!)

- set up your matrix augmented by the identity matrix

$$[B \mid \mathrm{I}] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

- use elementary row operations to get the augmented matrix into reduced row echelon form...

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \qquad -3r_1 + r_2 \to r_2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \qquad r_2 \div -2 \to r_2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right] \qquad -2r_2 + r_1 \to r_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right]$$

- this results in an augmented matrix of the form $\left[\mathrm{I} \mid B^{-1}\right]$, so ...

$$B^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix}$$

The same row operations that turned B into I, turned I into B^{-1}!

(A wonderful insomnia question: Why did that work?)

Using Matrix Inverses to Solve Systems of Equations

Why are matrix inverses important? One major use of matrix inverses is in solving systems of equations. Here's how:

Suppose we have the system of equations:

$$x + y = 7$$
$$x - y = 5$$

This is equivalent to the following matrix equation:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$ ← Try writing out this multiplication:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} =$$

Which can be summarized as:

$$A\,\vec{x} = \vec{b}$$

matrix

times an unknown vector

equals some known vector

Now, to solve this matrix equation, multiply on both sides (on the left) by A^{-1}:

$$A\,\vec{x} = \vec{b}$$

$$A^{-1} \cdot A\,\vec{x} = A^{-1} \cdot \vec{b}$$

$$\mathrm{I} \cdot \vec{x} = A^{-1} \cdot \vec{b}$$

$$\vec{x} = A^{-1} \cdot \vec{b}$$

So, the quick way to solve the system of equations (if we know the inverse) is to multiply the matrix inverse by the known vector. In this case:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$A \cdot \vec{x} = \vec{b}$$

$$\Rightarrow \vec{x} = A^{-1} \cdot \vec{b}$$

Aside:
Find the matrix inverse:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} .5 & .5 \\ .5 & -.5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 & .5 \\ .5 & -.5 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

done!

This method of solving systems of equations is particularly useful if you have many similar systems to solve, all with the same combinations of variables, but different totals on the right side:

$$\begin{aligned} x + y &= 17 \\ x - y &= 12 \end{aligned} \qquad \begin{aligned} x + y &= -3 \\ x - y &= 6 \end{aligned}$$

$$\begin{aligned} x + y &= 47.2 \\ x - y &= 19.7 \end{aligned} \qquad \begin{aligned} x + y &= ? \\ x - y &= ? \end{aligned}$$

etc...

This method is <u>not</u> useful if either the matrix inverse does not exist (which is quite common), and/or the system has 0 or ∞ solutions.

Section 4 -- Part 2

Solving systems of linear equations with 0, 1, or ∞ solutions.

You'll remember from earlier algebra classes that any system of linear equations might have 0, 1, or ∞ solutions. Let's review some of that material, re-examining it from the perspective of rref matrix solutions.

One-dimensional systems: 1 variable, usually with 1 equation.

$0x = 3 \quad \Rightarrow$ no solution, inconsistent

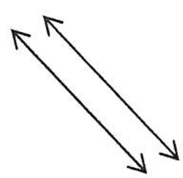
$5x = 15 \quad \Rightarrow$ one solution, consistent

$0x = 0 \quad \Rightarrow$ ∞ solutions, consistent and dependent

Two-dimensional systems: 2 variables, usually with 2 equations.

$$\begin{aligned} x+y&=3 \\ x+y&=4 \end{aligned}$$

no solution, inconsistent

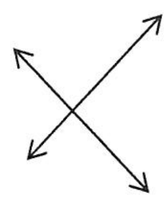


rref is:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\begin{aligned} x+y&=7 \\ x-y&=5 \end{aligned}$$

one solution, consistent



rref is:

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 1 \end{array}\right]$$

$$\begin{aligned} x+2y&=4 \\ 2x+4y&=8 \end{aligned}$$

∞ solutions, consistent and dependent

rref is:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

How do we describe the solutions of this last dependent system? As a vector equation of a line! More details soon.

More examples:

(Notice – you don't have to have the same number of equations as you do variables)

example 1: 0 solutions

$$\begin{aligned} x+y+z&=10\\ 2x+4y-z&=15\\ 2x+2y+2z&=30 \end{aligned}$$

$$\Rightarrow \text{ rref is } \left[\begin{array}{ccc|c} 1 & 0 & 2.5 & 0\\ 0 & 1 & -1.5 & 0\\ 0 & 0 & 0 & 1 \end{array}\right]$$

check out this last row:

$$0x+0y+0z=1$$

$$\Rightarrow 0=1$$

No way!

There is no solution, the system is inconsistent.

example 2: 1 unique solution

$$\begin{aligned} x+y&=7\\ x-y&=5\\ 3x+y&=19 \end{aligned}$$

$$\Rightarrow \text{ rref is } \left[\begin{array}{cc|c} 1 & 0 & 6\\ 0 & 1 & 1\\ 0 & 0 & 0 \end{array}\right]$$

what about this row?
$0x+0y=0$
is always true.
This row has no new information.

$$\Rightarrow \begin{aligned} x&=6\\ y&=1 \end{aligned} \quad \text{or} \quad \begin{bmatrix} x\\ y \end{bmatrix}=\begin{bmatrix} 6\\ 1 \end{bmatrix}$$

The system is consistent. (But not dependent)

example 3: infinite solutions:

$$\begin{aligned} x+y&=7 \\ 2x+2y&=14 \end{aligned}$$

$$\Rightarrow \text{ rref is } \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad x+y=7$$

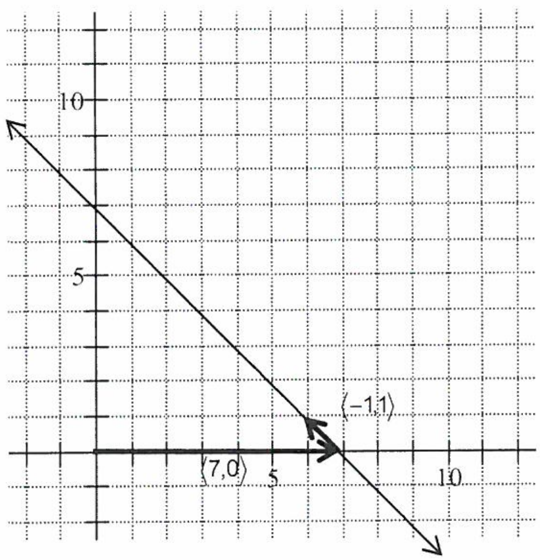
$$\Rightarrow \quad \begin{aligned} x&=7-y \\ y&=0+y \end{aligned}$$

add an extra equation so that there's one for each variable. Notice: $y = 0 + y$ is always true — that's why we can add it.

rewrite in vector form.

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which is the vector equation of a line!



Every point on this line solves the system of linear equations. This line can be described as a vector equation, or as the more familiar Cartesian equation. In this class, we'll prefer the vector equation.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad or \quad \langle x,y \rangle = \langle 7,0 \rangle + t\langle -1,1 \rangle \quad or \quad x+y=7$$

new form

old form

algebra I form.

example 3: infinite solutions:

$$\begin{aligned} x + y + z &= 10 \\ x \quad - z &= -6 \end{aligned}$$

$\Rightarrow$ rref is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & 1 & 2 & 16 \end{array}\right]$

$\Rightarrow$ simplified equations are $\begin{aligned} x - z &= -6 \\ y + 2z &= 16 \end{aligned}$

$\Rightarrow$ solve these equations for the leading variables. (z is the free variable, x and y depend on the value of z.)

$$\begin{aligned} x &= -6 + z \\ y &= 16 - 2z \\ z &= 0 + z \end{aligned}$$

add an extra equation that's true! so that there's one for each variable.

$\Rightarrow$ rewrite these equations in vector form to get your final answer:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 16 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

We could use z or t as the parameter here. When we use t, we're echoing our original vector descriptions of lines.

could also be written as $\langle x, y, z \rangle = \langle -6, 16, 0 \rangle + t \langle 1, -2, 1 \rangle$

What is this? It's the description of a line in 3-space ($\mathbb{R}^3$). Start at $\langle -6,16,0 \rangle$, and add any multiple of the vector $\langle 1,-2,1 \rangle$. Every point on this line solves the system:

$$\begin{aligned} x + y + z &= 10 \\ x \quad - z &= -6 \end{aligned}$$

To find particular solutions, let t equal any number and simplify:

t	$\langle -6,16,0 \rangle + t\langle 1,-2,1 \rangle$
-2	$\langle -8,20,-2 \rangle$
-1	$\langle -7,18,-1 \rangle$
0	$\langle -6,16,0 \rangle$
1	$\langle -5,14,1 \rangle$
2	$\langle -4,12,2 \rangle$

What would a graph of this solution look like?

example 4: infinite solutions:

$$\begin{array}{rrrrl} x_1 & -x_2 & & +2x_4 & =-1 \\ x_1 & +x_2 & +x_3 & -x_4 & = 2 \\ & -2x_2 & -x_3 & +3x_4 & =-3 \\ 5x_1 & -x_2 & +2x_3 & +4x_4 & = 1 \end{array}$$

$$\Rightarrow \text{ rref is } \left[\begin{array}{cccc|c} 1 & 0 & .5 & .5 & .5 \\ 0 & 1 & .5 & -1.5 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{lll} x_1 & +.5x_3 & +.5x_4 & =.5 \\ x_2 & +.5x_3 & -1.5x_4 & =1.5 \end{array}$$

$\Rightarrow$ solve for each variable in turn:

$$\begin{array}{lccc} x_1 = & .5 & -.5x_3 & -.5x_4 \\ x_2 = & 1.5 & -.5x_3 & +1.5x_4 \\ x_3 = & & x_3 & \\ x_4 = & & & x_4 \end{array}$$

add equations for x_3 and x_4 to complete the set

$\Rightarrow$ all solutions are of the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} .5 \\ 1.5 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -.5 \\ -.5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -.5 \\ 1.5 \\ 0 \\ 1 \end{bmatrix}$$

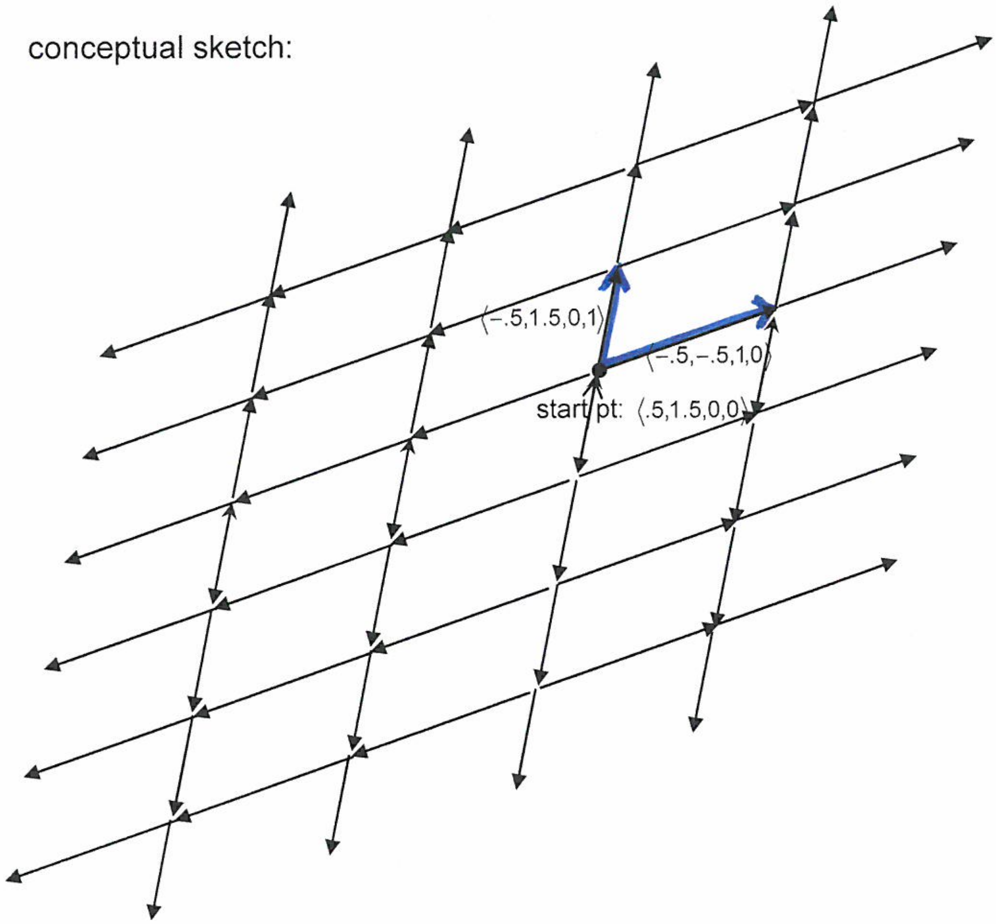
We're using s in place of x_3, and t in place of x_4.

which is a 2-dimensional plane in $\mathbb{R}^4$.

Some solutions:

	s = -2	s = -1	s = 0	s = 1	s = 2
t = -2					
t = -1					
t = 0			<.5,1.5,0,0>		
t = 1					
t = 2					

conceptual sketch:



The 2-d plane in $\mathbb{R}^4$ created from the starting vector $\langle .5,1.5,0,0\rangle$ plus any multiples of the two vectors $\langle -.5,-.5,1,0\rangle$ and $\langle -.5,1.5,0,1\rangle$. Every point on this plane solves the system!

cool!

not much

Summary:

Using rref to find the solutions of <u>any</u> system of linear equations.

To solve $A\vec{x} = \vec{b}$, we use the matrix $\left[A \mid b\right]$, which is A augmented by $\vec{b}$.

<u>no solutions</u> ≡ <u>inconsistent system</u>

rref is

$$\left[\begin{array}{cccc|c} & & & & \\ 0 & 0 & \cdots & 0 & 1 \\ & & & & \end{array}\right]$$

this row shows there is no solution, since 0 = 1 is not true.

<u>one solution</u> ≡ <u>consistent system</u>

rref is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \# \\ 0 & 1 & 0 & 0 & \# \\ 0 & 0 & 1 & 0 & \# \\ 0 & 0 & 0 & 1 & \# \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

one unique sol'n iff there are exactly as many independent eq'ns as variables.

you might have extra rows of zeros -- that's ok.

<u>infinite solutions</u> ≡ <u>dependent system</u>

$$\text{rref is } \left[\begin{array}{ccc|cc|c} 1 & 0 & 0 & \# & \# & \# \\ 0 & 1 & 0 & \# & \# & \# \\ 0 & 0 & 1 & \# & \# & \# \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

dependent iff there are more variables than independent equations.

extra rows of zeros are ok

ind. eq'ns

dim of sol'n space

total # variables

$$\Rightarrow \begin{array}{c}\text{dimension of}\\ \text{sol'n space}\end{array} = \begin{array}{c}\text{total \#}\\ \text{variables}\end{array} - \begin{array}{c}\text{\# independent}\\ \text{equations (rank}\end{array}$$

<u>Definition</u>:
rank ≡ number of non-zero rows in rref

$$rank(A \mid b) > rank(A)$$
$$\Leftrightarrow$$
no solution

$$rank(A \mid b) = \#\text{ of variables}$$
$$= \#\text{ of columns in A}$$
$$\Leftrightarrow$$
one unique solution

$$rank(A \mid b) < \#\text{ of variables}$$
OR
$$rank(A \mid b) < \#\text{ of columns in A}$$
$$\Leftrightarrow$$
infinite solutions

One last calculator trick:

If your matrix has more rows than columns, the TI-83 doesn't know how to get reduced row echelon form (rref).

Try it:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

ask calculator for rref?

ERR: INVALID DIM :(

Here's how to make it work:

- add extra columns of all zeros to your matrix until the number of rows equals the number of columns:
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{bmatrix}$$
- ask for rref:
$$rref([A]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
- strip off the extra columns of zeros, write down the answer:
$$rref\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad :)$$

Section 5: Linear Transformations

BIG IDEA:

Multiplication by a matrix with m rows and n columns is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$\underset{\text{(domain)}}{\mathbb{R}^n} \xrightarrow{\text{mult by } A_{mxn}} \underset{\text{(range)}}{\mathbb{R}^m}$$

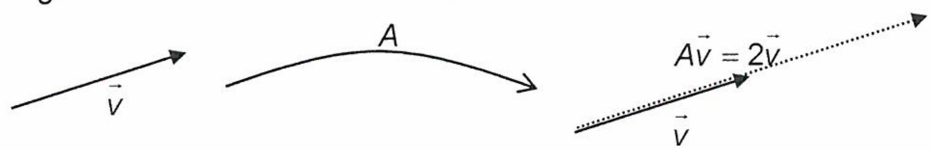
Notation:
$A_{m \times n}$ is a matrix with $\underline{m}$ rows, and $\underline{n}$ columns.

a particular n-vector $\xmapsto{\text{mult by } A_{m\times n}}$ a particular m-vector

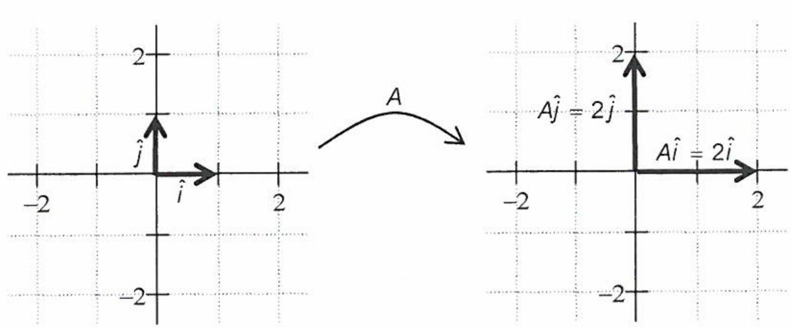
<u>an example (from Lab ~~8~~ B)</u>

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Multiplication by A is a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. It takes every vector in $\mathbb{R}^2$ and doubles its length but leaves the direction the same:



Also:



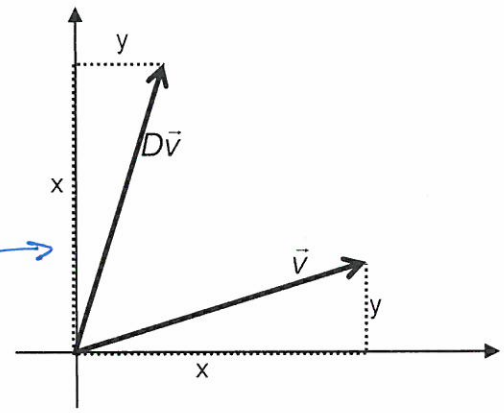
<u>another example (from Lab ~~8~~ B)</u>

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

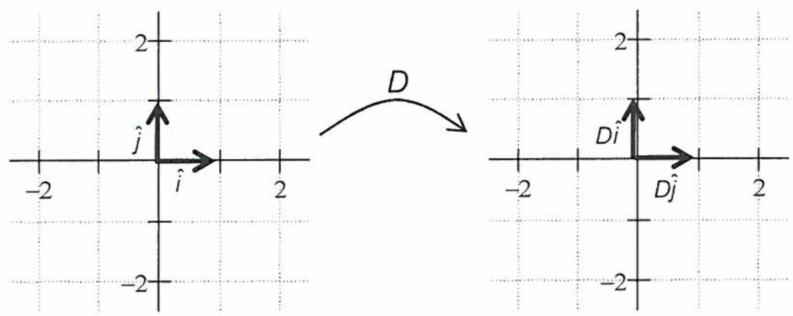
Multiplication by D is a <u>function</u> from $\mathbb{R}^2$ to $\mathbb{R}^2$. It takes every vector in $\mathbb{R}^2$ and switches its horizontal and vertical components:

$$\vec{w} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \Rightarrow D\vec{w} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$D\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \longrightarrow$$



What happens to $\hat{i}$ and $\hat{j}$?



...$\hat{i}$ and $\hat{j}$ switch!

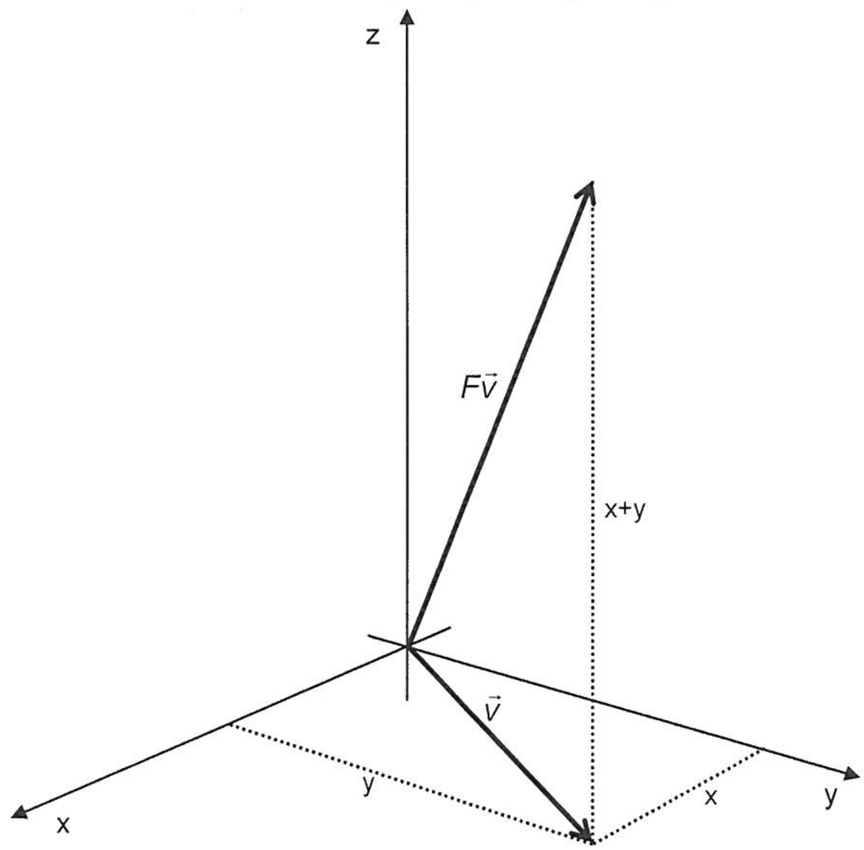
What is D^{-1}? Why does that make sense?

another example:

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Multiplication by F is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$. Can you figure out what it does?

any old vector in $\mathbb{R}^2$ $= \begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{F} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$



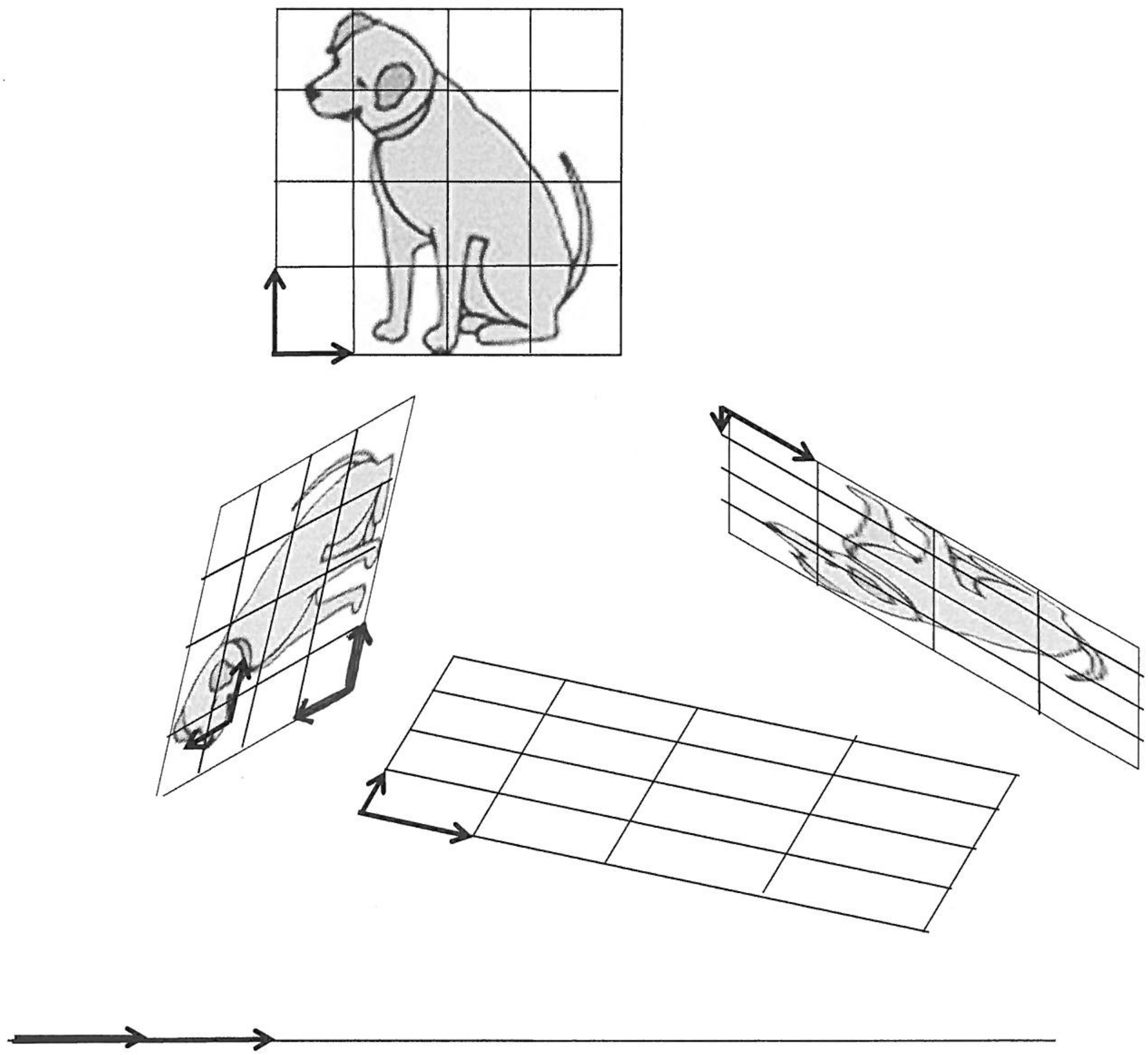
What happens to $\hat{i}$ and $\hat{j}$?

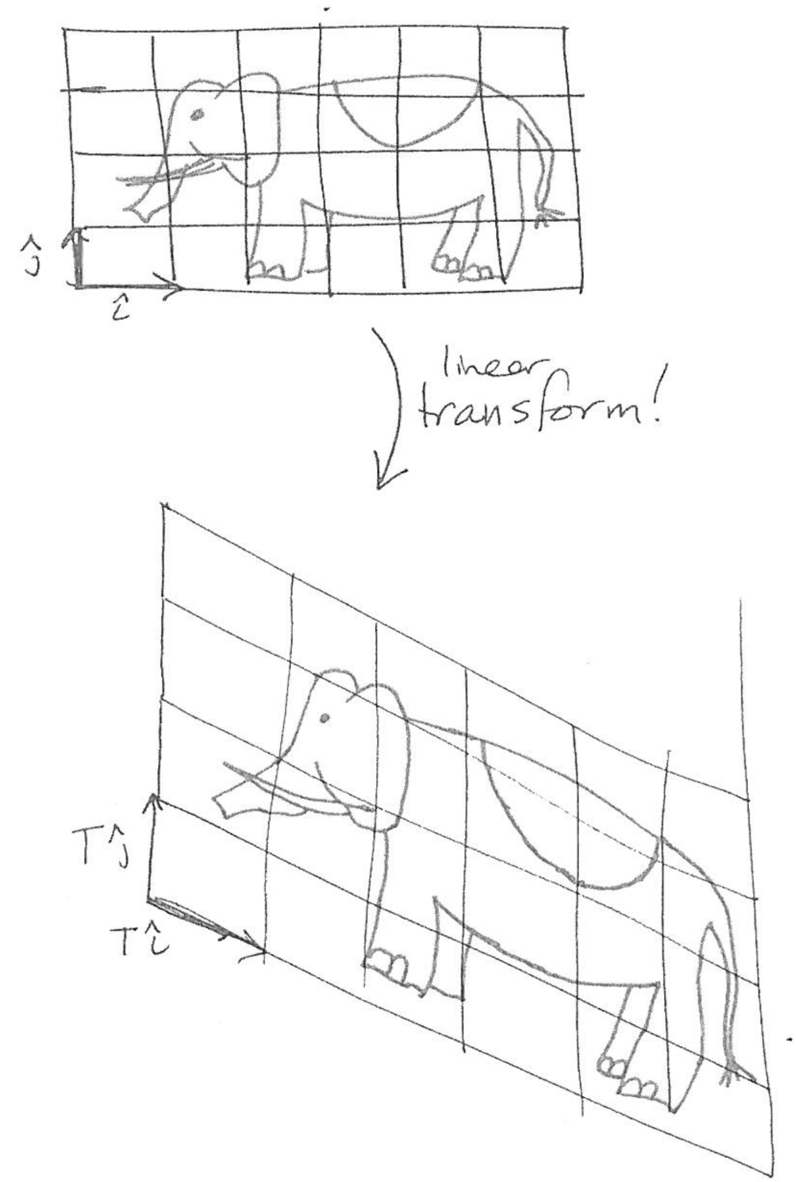
Math 253 Activity

Some examples of linear transformations:

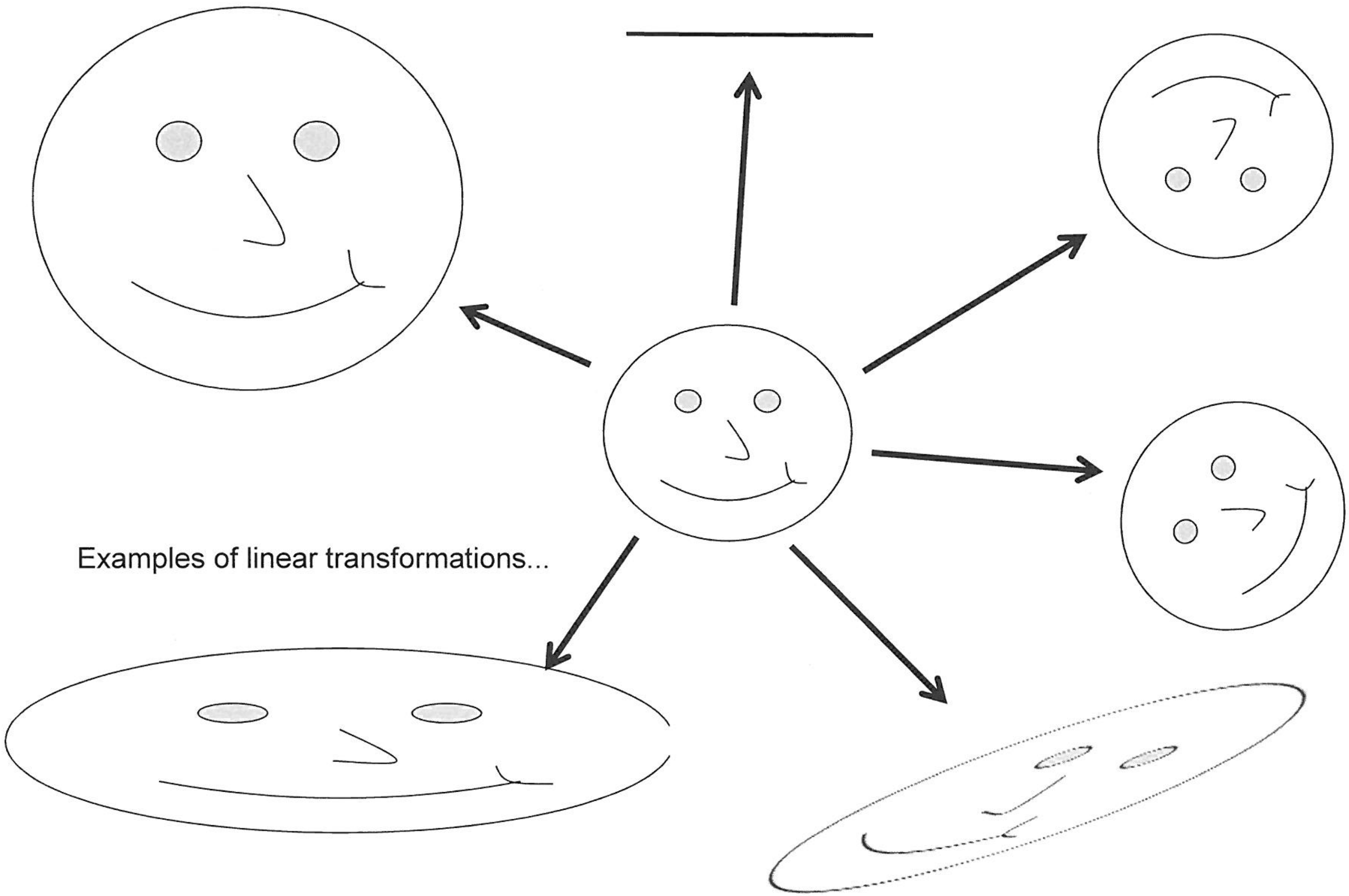
"If we know where the basis (building block) vectors go, then we know where <u>every</u> vector (every "point") in the domain (universe) goes."

Notice: Straight lines stay straight.
Parallel lines stay parallel.
The origin stays in the same place.




$\hat{j}$
$\hat{\imath}$
linear transform!
$T\hat{j}$
$T\hat{\imath}$

Examples of linear transformations...



The functions "multiply by a matrix" have some special properties that make them linear transformations.

Properties of Linear Transformations

$f(a+b) = f(a) + f(b)$ ← a linear transformation distributes over addition

$f(k \cdot a) = k \cdot f(a)$ ← a linear transformation commutes with multiplication by a constant.

if k is any number.

A Useful Fact About Linear Transformations:

$f(k \cdot a) = k \cdot f(a)$

let k = 0

$f(0 \cdot a) = 0 \cdot f(a)$

$\boxed{f(0) = 0}$

A linear transformation always maps 0 to 0. (The origin never moves.)

Why is it important that matrix multiplication is a linear transformation? Because now if we know what happens to $\hat{i}$ and $\hat{j}$, we know what happens to every vector!

Suppose $\vec{v} = c\hat{i} + d\hat{j}$. Then $A\vec{v} = A\left(c\hat{i} + d\hat{j}\right)$

$= A(c\hat{i}) + A(d\hat{j})$ by property 1

$= c(A\hat{i}) + d(A\hat{j})$ by property 2

(see the picture on page ~~87!~~ 84!)

$\hat{i}$ and $\hat{j}$ make a grid – when you do the transformation, moving $\hat{i}$ and $\hat{j}$ changes the whole grid, and everything on it.

Question? How many functions (that are not matrix multiplication) can you think of that satisfy these properties? Anything from Math 111? Anything from trig?[3]

Aside: It can be shown that any function from $\mathbb{R}^n$ to $\mathbb{R}^m$ that satisfies the linear transformation properties can be written as matrix multiplication, but that's beyond the scope of this course.

[3] There's only ONE such function: f(x) = mx (where m is any number) only! In fact, this function is one-dimensional matrix multiplication, where the matrix = $[m]$.

not math

Let's check that matrix multiplication satisfies the linear transformation properties -- try it out with the matrix $B=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$:

property 1: $f(a+b)=f(a)+f(b)$

our example:

$$B(\vec{v}_1+\vec{v}_2)=B(\vec{v}_1)+B(\vec{v}_2)$$

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\left(\begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix}+\begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix}\right)=$$

add the vectors

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}x_1+x_2\\ y_1+y_2\\ z_1+z_2\end{bmatrix}=$$

multiply

$$\begin{bmatrix}1(x_1+x_2)+2(y_1+y_2)+3(z_1+z_2)\\ 4(x_1+x_2)+5(y_1+y_2)+6(z_1+z_2)\end{bmatrix}=$$

distribute

$$\begin{bmatrix}1x_1+1x_2+2y_1+2y_2+3z_1+3z_2\\ 4x_1+4x_2+5y_1+5y_2+6z_1+6z_2\end{bmatrix}=$$

rearrange terms

$$\begin{bmatrix}1x_1+2y_1+3z_1+1x_2+2y_2+3z_2\\ 4x_1+5y_1+6z_1+4x_2+5y_2+6z_2\end{bmatrix}=$$

separate matrices

$$\begin{bmatrix}1x_1+2y_1+3z_1\\ 4x_1+5y_1+6z_1\end{bmatrix}+\begin{bmatrix}1x_2+2y_2+3z_2\\ 4x_2+5y_2+6z_2\end{bmatrix}=$$

"un-mulitply"

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix}=$$

$$B(\vec{v}_1)+B(\vec{v}_2)=B(\vec{v}_1)+B(\vec{v}_2)$$ yes!

(aside: why did I choose 3-d vectors as inputs? what dimension are the output vectors?)

property 2: $f(k\cdot a)=k\cdot f(a)$

where k is any number

$$B(k\cdot\vec{v}_1)=k\cdot B(\vec{v}_1)$$

our example: $\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\begin{bmatrix}k\cdot x_1\\ k\cdot y_1\\ k\cdot z_1\end{bmatrix}=$

then what?

... ok, that was a lot of abstraction.   Let's do something more concrete ...

The Determinant of a Matrix

For a 2 x 2 matrix:

algebraically:

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $|A| = \det(A) = \underset{\substack{\text{product of}\\ \text{main diagonal}}}{ad} \underset{\text{minus}}{-} \underset{\substack{\text{product of}\\ \text{off diagonal}}}{bc}$ is the determinant of matrix.

geometrically:

Think of your matrix as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$.   Find $A\hat{i}$ and $A\hat{j}$[4], and find the area of their vector parallelogram.

- the area is positive if the movement from $A\hat{i}$ and $A\hat{j}$ is still counterclockwise.
- the area is negative if the movement $A\hat{i}$ and $A\hat{j}$ changes to clockwise.
- the area is zero is $A\hat{i}$ and $A\hat{j}$ are colinear (parallel).

insomnia question: Why are these two definitions the same?

[4] These are called the "images" of $\hat{i}$ and $\hat{j}$ under the transformation A.   An image is the output for any particular input to a function.   If multiplication by the matrix A is the function, and $\hat{i}$ is the input, then $A\hat{i}$ is the image of $\hat{i}$ under A.

Some 2 x 2 examples of determinants:

<u>Notice:</u>

$$F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

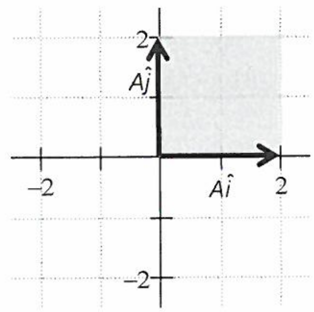
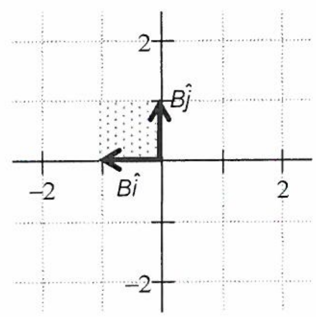
$$\Rightarrow \quad F\hat{i} = \begin{bmatrix} a \\ c \end{bmatrix}$$

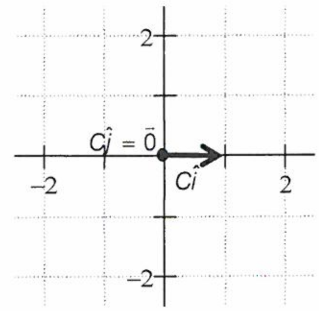
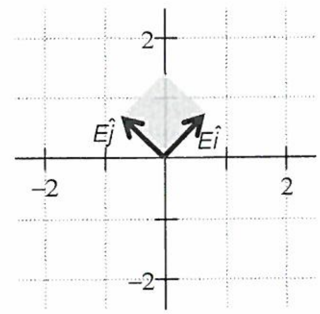
and $F\hat{j} = \begin{bmatrix} b \\ d \end{bmatrix}$, so...

$F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

the first column of the matrix will be $F\hat{i}$.

the second column of the matrix will be $F\hat{j}$.

matrix	algebra (arithmetic)	geometry
$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $A\hat{i}$ $A\hat{j}$	$\lvert A\rvert = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$ $= (2)(2) - (0)(0)$ $= 4$	 area of the vector parallelogram is 4.
$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $B\hat{i} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $B\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\lvert B\rvert = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$ $= (-1)(1) - (0)(0)$ $= -1$	 area is -1. (negative because $B\hat{j}$ is clockwise from $B\hat{i}$.)

$C=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$	$\lvert C\rvert=\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix} = (1)(0)-(0)(0) = 0$	 area is 0.
$E=\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{bmatrix}$	$\lvert E\rvert=\begin{vmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{vmatrix} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{-\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{2}{4}-\frac{-2}{4} = \frac{1}{2}+\frac{1}{2} = 1$	 area is 1. The unit square rotated 45°.

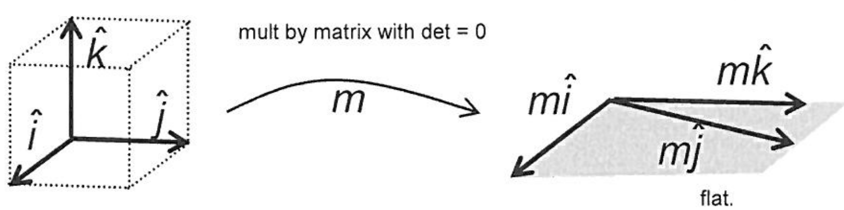
One last way to think of determinants: How much does the matrix increase or decrease all areas? For matrix A above, every area increases by 4 times (because the length of every vector increases 2 times).

Determinants in Higher Dimensions

Notice that books and websites talk a lot about calculating determinants, and not so much about what they mean. We're interested in their meaning and uses, and are usually content to let the calculator do the arithmetic. On the TI-83: 2nd Matrix MATH 1: det(
On the TI-89: 2nd Math Matrix det

<u>Here's the important stuff:</u>

- Determinants are only defined for <u>square</u> matrices.
- An n x n matrix's determinant is the <u>n-dimensional volume</u> of the n-dimensional parallelopiped formed by $M\hat{i}$, $M\hat{j}$, $M\hat{k}$, etc... (the images under matrix multiplication of $\hat{i}$, $\hat{j}$, $\hat{k}$, etc...)
- The determinant is also the factor by which "n-volumes" in $\mathbb{R}^n$ are stretched or shrunk in the transformations caused by multiplying by the matrix. It will be negative if the order of the vectors switches from right-handed to left-handed (if the volume is turned inside out).
- If the determinant is zero, then at least one dimension has been smushed out in the transformation. For example, in $\mathbb{R}^3$, if the images of $\hat{i}$, $\hat{j}$, and $\hat{k}$ are coplanar, then the volume of their parallelopiped is 0.



Calculating 1 x 1, 2 x 2, 3 x 3 determinants

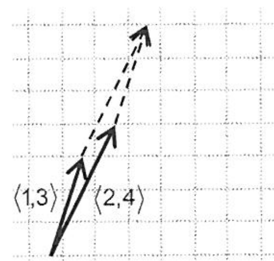
<u>1 x 1:</u>

$$A = [5] \Rightarrow \det(A) = 5$$

(by the way, what function is this? It's f(x)=5x.)

<u>2 x 2:</u>

<u>insomnia</u>: Why does this give the area of the parallologram?



$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \underset{-\qquad +}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$$

$$\Rightarrow (1)(4) - (2)(3) = -2$$

<u>3 x 3:</u>

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Write $[C \mid C]$

$$\Rightarrow \underset{-\quad -\quad -\quad +\quad +\quad +}{\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 \end{array}\right]}$$

just like cross-product!

multiply the numbers on the right-hand diagonals, count them as positive, add them up.
multiply the numbers on the left hand diagonals, count them as negative, add them up also.

total = determinant = (45 + 84 + 96) - (48 + 72 + 105) = 0

$$|C| = 0$$

<u>4 x 4 and n x n:</u>

Unfortunately, this method does not work. Ask your calculator instead!

Some properties of determinants

Think about each of these properties in terms of our geometric definition of determinant as the number by which "n-volumes" are multiplied when multiplication by the matrix maps $\mathbb{R}^n$ to $\mathbb{R}^n$.

(1) $|A| = |A^T|$ or $\det(A) = \det(A^T)$

(2) If you have a row or column of all 0's, then the determinant is 0.

(3) Adding a multiple of one row to another row does not change the determinant.

(4) Switching two rows (or columns) switches the sign of the determinant.

(5) Multiplying a row (or column) by a constant multiplies the determinant by that constant.

(6) $\det(A \cdot B) = \det(A) \cdot \det(B)$

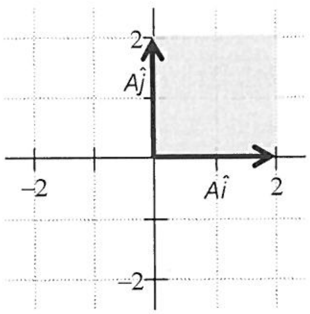
Why do these make sense?

Play with them!

Which matrices have inverses?

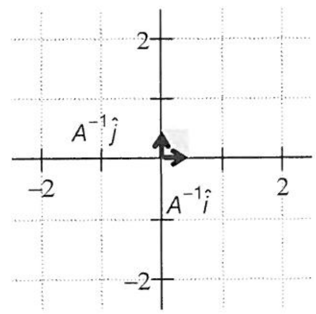
Let's go back to a question we were working on -- which matrices have inverses, and which ones don't? Let's look at some examples, remembering to think of matrices as defining functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. (Only square matrices have a chance of having an inverse.)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



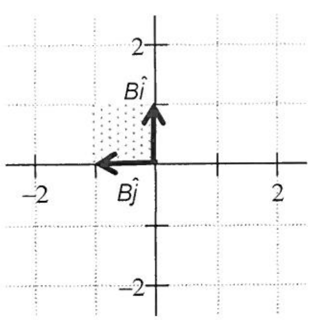
$\det(A) = 4$

$$A^{-1} = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$$



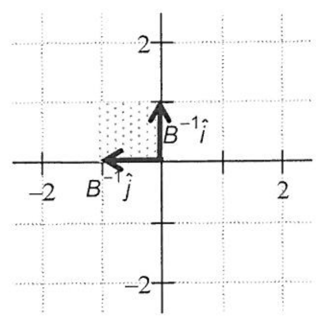
$\det(A^{-1}) = \frac{1}{4}$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

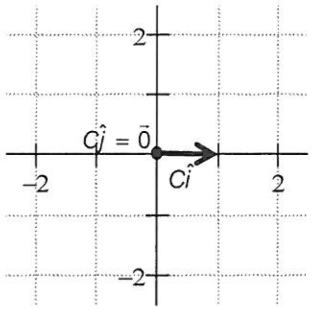
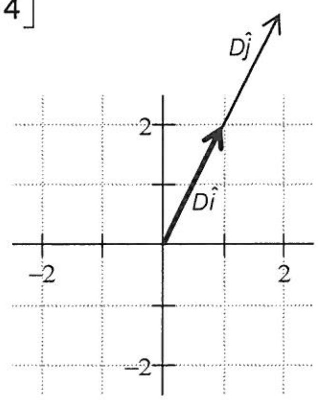


$\det(B) = -1$

$$B^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



$\det(B^{-1}) = -1$

$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  $\det(C) = 0$	No Inverse. Why not?
$D = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  $\det(D) = 0$	No Inverse. Why not?

Conclusions for 2 x 2 matrices:

A matrix M has an inverse

iff

$\det(M) \neq 0$

iff

(i.e. $M\hat{i}$ and $M\hat{j}$)

The images of $\hat{i}$ and $\hat{j}$ under multiplication by M are not parallel (define a non-zero area parallelogram).

iff

One row (or column) is not a multiple of the other.

Aside: Did you notice? Determinant of an inverse is the reciprocal of the determinant of the matrix.

$$|A^{-1}| = \frac{1}{|A|}$$

Why does that make sense?

How does that help us remember that if the determinant is zero, then the inverse doesn't exist?

Conclusions For Higher Dimensions:

Remember: Any m x n matrix is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Matrix M is invertible (has an inverse)

iff

m = n (the matrix is square), and $\det(M) \neq 0$

iff

m = n, and the images of $\hat{i}$, $\hat{j}$, $\hat{k}$, (i.e., $M\hat{i}$, $M\hat{j}$, $M\hat{k}$, etc...) etc... (the basis vectors) under multiplication by M are independent vectors.

iff

m = n, and the "n-volume" of the parallelopiped formed by $M\hat{i}$, $M\hat{j}$, $M\hat{k}$, etc ... is non-zero.

Definition: A matrix with no inverse is called singular.

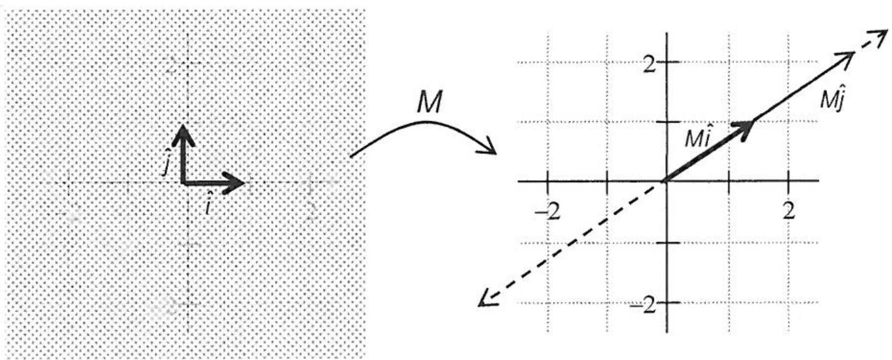
Have I said this explicitly? If multiplication by A is a function, then multiplication by A^{-1} is the inverse function.

Why does this make sense?

If a 2 x 2 matrix transforms two non-parallel vectors (like $\hat{i}$ and $\hat{j}$) into two parallel vectors, you've <u>lost</u> information -- the function is <u>not one-to-one</u>, and therefore has <u>no inverse</u>.

A typical non-invertible matrix multiplication:

$$\underset{\text{(domain)}}{\mathbb{R}^2} \xrightarrow{\text{mult by M}} \underset{\text{(range)}}{\mathbb{R}^2}$$



all combinations of $\hat{i}$ and $\hat{j}$ cover the plane

all combinations of $M\hat{i}$ and $M\hat{j}$ only cover the line

$\Rightarrow$ <u>not</u> invertible

What about higher dimensions?

The same idea applies -- if the determinant is 0, the matrix multiplication function from $\mathbb{R}^n$ to $\mathbb{R}^n$ "smushes" out some information. The n-dimensional cube formed by $\hat{i}$, $\hat{j}$, $\hat{k}$, etc... gets flattened so that the volume formed by $M\hat{i}$, $M\hat{j}$, $M\hat{k}$, etc... is zero.

Therefore, you've lost information, the function is <u>not one-to-one</u>, and is <u>not invertible</u>.

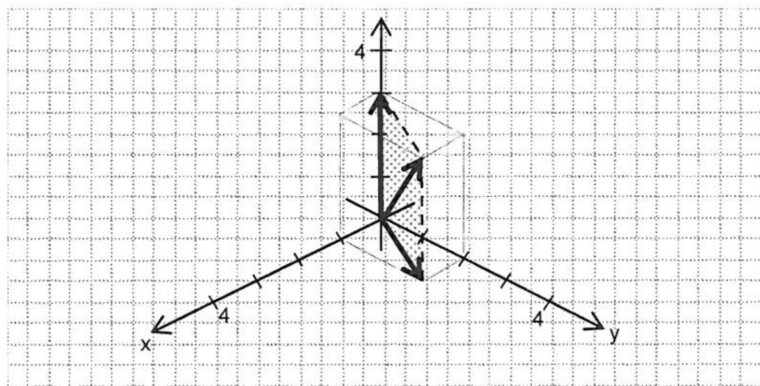
<u>typical example:</u>

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix} \quad \text{is a function from } \mathbb{R}^3 \text{ to } \mathbb{R}^3.$$

$$A\hat{i} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

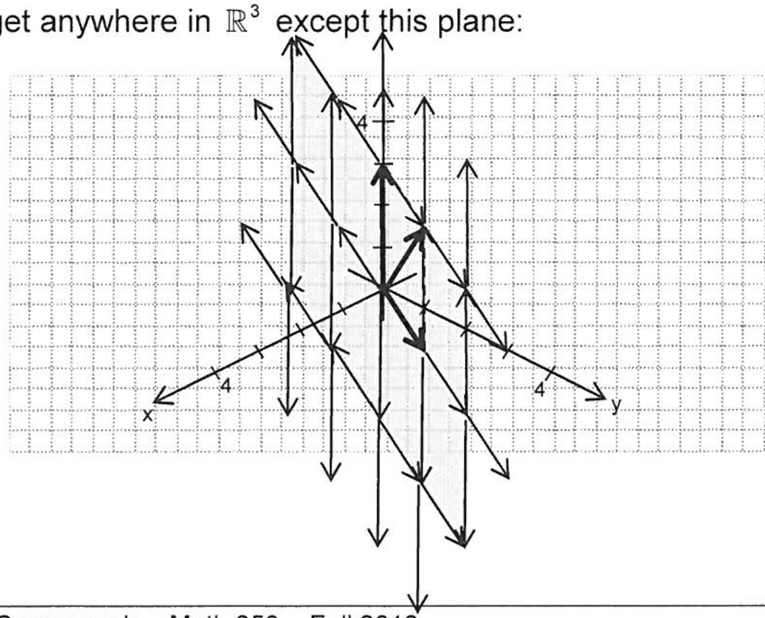
$$A\hat{j} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$A\hat{k} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



uh-oh! they're all on the same plane -- so the volume they form is 0.

Also, what's the complete image of $\mathbb{R}^3$? Just the plane formed by these three vectors ... you can't get anywhere in $\mathbb{R}^3$ except this plane:



Section 6: Solutions of Systems of Linear Equations Explained By Matrix Multiplication Functions.

Remember, any system of linear equations can be written as:

$$A\,\vec{x} = \vec{b}$$

matrix

times an unknown vector

equals some known vector

Or, using the idea of functions:

$$\vec{x} \xmapsto{\quad A \quad} \vec{b}$$

unknown vector

transformed by matrix multiplication

becomes a known vector

<u>So</u>, the question of finding the solutions to a system of linear equations becomes the question of finding all the vectors $\vec{x}$ that are transformed by the matrix A into the vector $\vec{b}$.

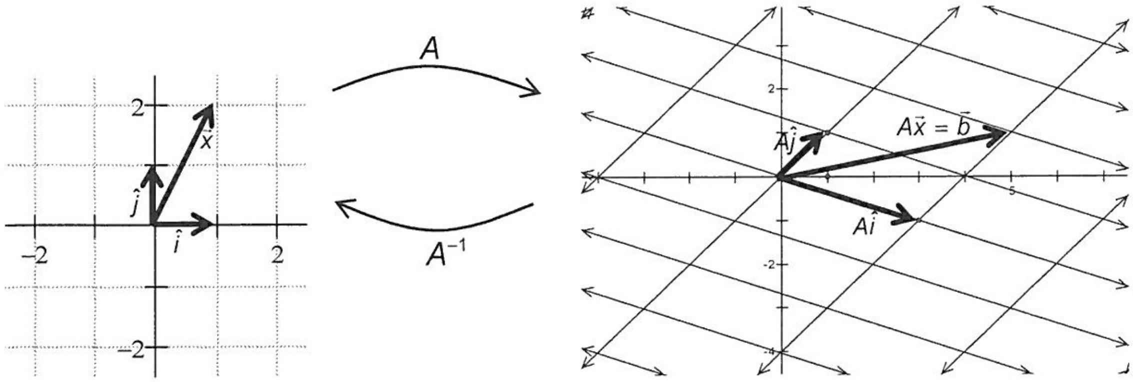
Let's go through the possible cases with 2 equations and 2 variables -- in other words, the possible transformations caused by 2 x 2 matrices:

<u>Case 1</u>: One unique solution iff A is invertible (if $\det(A) \neq 0$).

<u>Case 2</u>: No solution at all (might happen if det(A)=0).

<u>Case 3</u>: Infinite solutions (might happen if det(A)=0).

Case 1: One unique solution iff A is invertible (if $\det(A) \neq 0$).



The transformation of the domain $\mathbb{R}^2$ by A hits every point in the range $\mathbb{R}^2$, and hits each point exactly once (because it's a linear transformation), so there is some unique vector $\vec{x}$ that was mapped to $\vec{b}$.

$$\boxed{\vec{x}=A^{-1}\vec{b}}$$

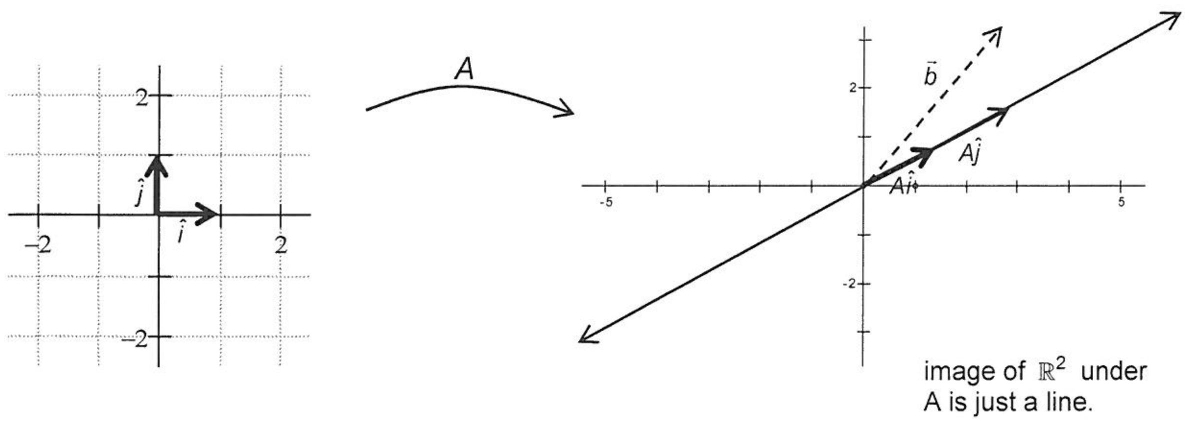
Cool Observation: if $\vec{b} = 1(A\hat{i}) + 2(A\hat{j})$
then $\vec{x} = 1(\hat{i}) + 2(\hat{j})$.

This is a direct consequence of the properties of linear transformations! How?

Because:

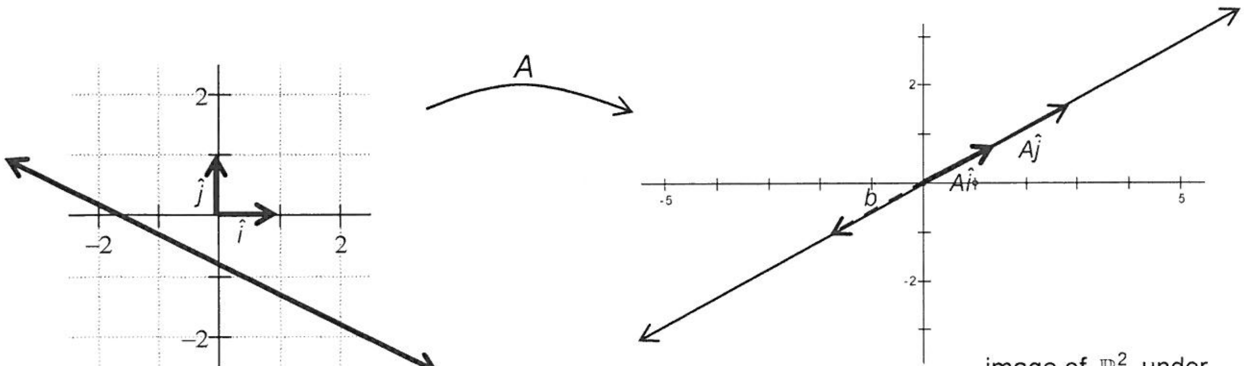
$$\begin{aligned} A\vec{x} &= A\left(1(\hat{i}) + 2(\hat{j})\right) \\ &= A(\hat{i} + 2\hat{j}) \\ &= A\hat{i} + 2A\hat{j} \\ &= 1(A\hat{i}) + 2(A\hat{j}) \\ &= \vec{b} \end{aligned}$$

Case 2: No solution at all (might happen if det(A)=0)



uh-oh! Vector $\vec{b}$ isn't in the image of $\mathbb{R}^2$ under A -- it never got hit -- there is no $\vec{x}$ such that $A\vec{x} = \vec{b}$.

Case 3: Infinite solutions (might happen if det(A)=0)



$\vec{x}$ can be any vector on some line.

image of $\mathbb{R}^2$ under A is just a line.

As $\mathbb{R}^2$ was smushed by A down onto a line (as 2 dimensions went down to 1), an infinite number of points $\vec{x}$ were mapped onto the single point $\vec{b}$.

$$\boxed{A\vec{x} = \vec{b} \text{ has infinite solutions.}}$$

Similar descriptions apply in higher dimensions (if we still assume that A is square, that m = n). If the function A is one-to-one, then $A\vec{x} = \vec{b}$ has one unique solution.

If the function A is many-to-one instead, you'll have 0 solutions if $\vec{b}$ is not in the image of $\mathbb{R}^n$ under A, infinite solutions if $\vec{b}$ is in the image. The dimension of the solution space will be the same as the number of dimensions lost (smushed out) in the transformation.

not much

Homogenous Systems

$$A\,\vec{x}=\vec{0} \qquad \underline{\text{for example:}} \qquad \begin{aligned} 2x-3y&=0 \\ 7x+2y&=0 \end{aligned}$$

Homogenous systems are systems where all the equations equal zero. They have some interesting properties:

(1) $A\,\vec{x}=\vec{0}$ always has at least one solution (because it is a linear transformation, so $\vec{x}=\vec{0}$ is always a solution).

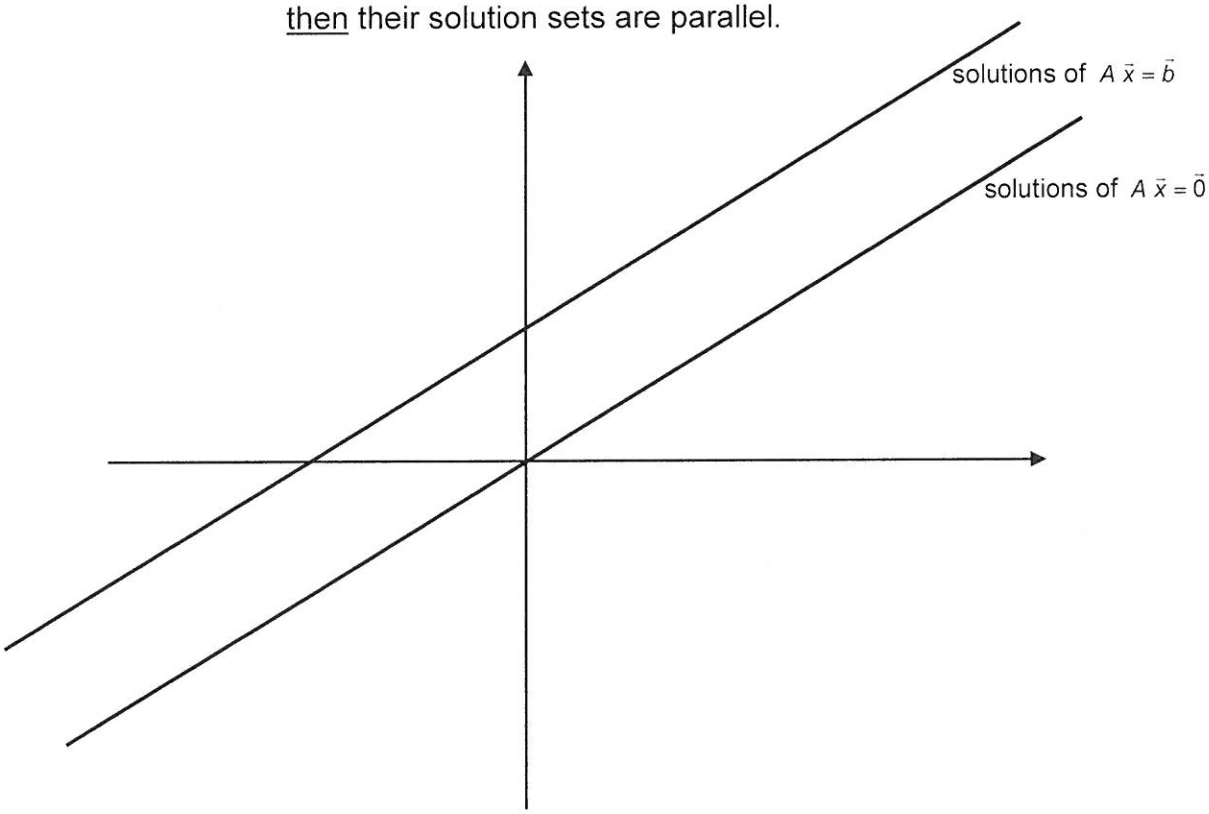
(2) If A is one-to-one (if $\det(A)\neq 0$, etc...) then $\vec{x}=\vec{0}$ is the only solution of $A\,\vec{x}=\vec{0}$.

(3) If A is many-to-one (if $\det(A)=0$), then $A\,\vec{x}=\vec{0}$ has infinite solutions

AND

if $A\,\vec{x}=\vec{b}$ has infinite solutions

<u>then</u> their solution sets are parallel.



Appendix: One last formula for the inverse of a 2 x 2 matrix.

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$=\frac{1}{\det(A)}\begin{bmatrix} \text{matrix A with main diagonal entries switched,} \\ \text{and off-diagonal entries multiplied by -1} \end{bmatrix}$$

(Notice: Yet one more reflection of a basic fact:
A^{-1} exists if and only if $\det(A)\neq 0$!)

example:

$$B=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow \qquad B^{-1}=\frac{1}{(1)(4)-(2)(3)}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$=\frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

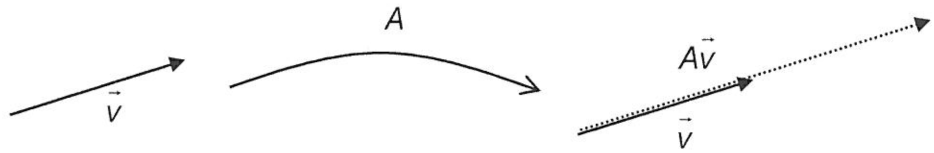
$$=\begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix}$$

:)

Section 7: Eigenvalues and Eigenvectors.

Suppose that A is an n x n matrix, so that A is also a function (specifically, a linear transformation) from $\mathbb{R}^n$ to $\mathbb{R}^n$.

- an <u>eigenvector</u> of A is any vector $\vec{v}$ in $\mathbb{R}^n$ such that $A\vec{v}$ is parallel to $\vec{v}$.

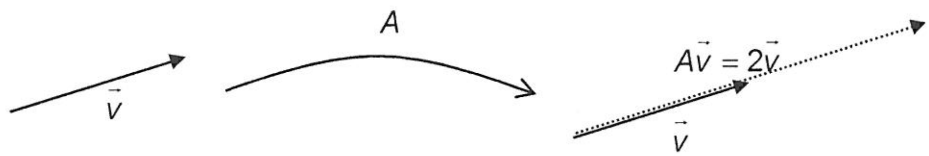


- An <u>eigenvalue</u> of a matrix is a number λ (lambda) such that $A\vec{v} = \lambda\vec{v}$ for some eigenvector $\vec{v}$ of A.

<u>Example 1:</u>

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

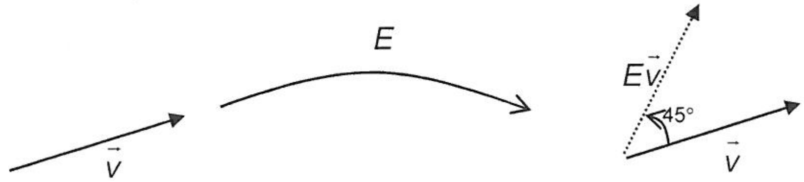
The matrix A stretches every vector in $\mathbb{R}^2$ by 2. Therefore, <u>every</u> vector in $\mathbb{R}^2$ is an eigenvector of A, and the corresponding eigenvalue is 2.



<u>Example 2:</u>

$$E = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

The matrix E rotates every vector in $\mathbb{R}^2$ by 45° counterclockwise. Therefore, <u>no</u> vector in $\mathbb{R}^2$ ends up parallel to itself, and E has <u>no eigenvectors nor eigenvalues.</u>

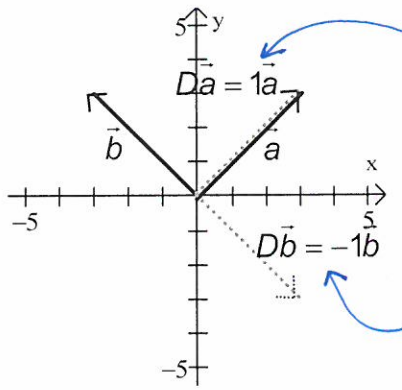


Example 3:

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The matrix D reflects every vector in $\mathbb{R}^2$ across the line y = x. Therefore, any vector on the line y = x stays the same -- so it is an eigenvector of D whose corresponding eigenvalue is 1.

Any vector on the line y = -x is turned exactly backwards, so any such vector is an eigenvector whose eigenvalue is -1.



if $\vec{a} = \begin{bmatrix} t \\ t \end{bmatrix}$ for any t

if $\vec{b} = \begin{bmatrix} -t \\ t \end{bmatrix}$ for any t

But, how do you find eigenvalues and eigenvectors in general?

Let v be an eigenvector, and λ its corresponding eigenvalue for a given matrix A. Then ...

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ \Rightarrow A\vec{v} - \lambda\vec{v} &= \vec{0} \\ \Rightarrow (A - \lambda \mathrm{I})\vec{v} &= \vec{0} \end{aligned}$$

matrix·vector=zero vector

why does I have to be there?

This is now a system of linear equations, and it only has non-zero solutions if the determinant of the matrix $(A - \lambda \mathrm{I})$ is 0 (see homogenous systems in section 6).

$\Rightarrow$ $\det(A - \lambda \mathrm{I}) = 0$ if and only if λ is an eigenvalue of A.

Therefore, here's THE PLAN:

- find $\det(A - \lambda \mathrm{I})$, set it equal to 0.
- this gives you a polynomial equation with the variable λ, called the characteristic polynomial. Find all possible solutions for λ -- these will be the eigenvalues.
- plug each eigenvalue (λ) into the matrix equation/linear system $(A - \lambda \mathrm{I})\vec{x} = \vec{0}$ and find all the non-zero values for $\vec{x}$. These are the eigenvectors.

Let's try it!

Back to example 3:

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Step 1: Find the characteristic polynomial.

$$\det(D - \lambda \mathrm{I}) = 0$$

$$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$(-\lambda)(-\lambda) - (1)(1) = 0$$

$$\lambda^2 - 1 = 0$$

Step 2: Solve for λ.

$$\lambda^2 - 1 = 0$$
$$(\lambda - 1)(\lambda + 1) = 0$$
$$\lambda - 1 = 0 \quad or \quad \lambda + 1 = 0$$
$$\lambda = 1 \quad or \quad \lambda = -1$$

These are the eigenvalues!

Step 3: Find the corresponding eigenvectors.

Let $\lambda = 1$:

$$(D - \lambda \mathrm{I})\vec{x} = \vec{0}$$

$$\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$rref:$$

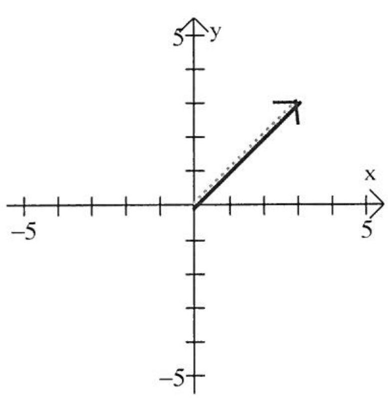
$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = 0 + y$$
$$y = 0 + y$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Any vector of the form $\begin{bmatrix} t \\ t \end{bmatrix}$ is an eigenvector of D with a corresponding eigenvalue 1. Remember what this means -- if you have a vector of the form $\begin{bmatrix} t \\ t \end{bmatrix}$, then the transformation D doesn't change it.



$$D\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Let $\lambda = -1$:

$$(D - \lambda \mathrm{I})\vec{x} = \vec{0}$$

$$\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \vec{x} = \vec{0}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$rref:$$

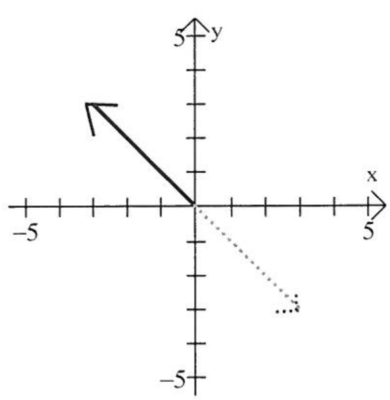
$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x + y = 0$$

$$\Rightarrow x = 0 - y$$
$$y = 0 + y$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Any vector of the form $\begin{bmatrix} -t \\ t \end{bmatrix}$ is an eigenvector of D with a corresponding eigenvalue of -1. Remember what this means -- if you have a vector of the form $\begin{bmatrix} -t \\ t \end{bmatrix}$, then the transformation D multiplies it by -1 (turns it around).



$$D\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix}$$

Another example of finding eigenvalues:

$$G = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\left| \begin{bmatrix} 1-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix} \right| = 0$$

$$(1-\lambda)^2 - 0 = 0$$

$$(1-\lambda)^2 = 0$$

$$1-\lambda = 0$$

$$1 = \lambda$$

... is the eigenvalue of matrix G. Can you find the corresponding eigenvector?

$$(G-\lambda \mathrm{I})\vec{x} = \vec{0}?$$

$$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

rref = ?

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

or, in other words, y = 0 and x = anything, so...

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Therefore, any vector on the x-axis stays the same, no other vector is parallel to itself. See if you can make sense of this by graphing the transformation, graphing $G\hat{i}$ and $G\hat{j}$ ~~(compare to HW 22 Problem 1).~~